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Line Congruences.

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This subject, which has already been otherwise treated by Hamilton and Kummer (Crelle, Vol. LVII), I propose to treat by quaternions. I have obtained all the results given by Kummer in the above-mentioned paper, and have also been enabled by the method employed to carry out certain steps which he has only indicated. Since in this paper the straight line is determined as a function of two independent parameters, there will be a double infinity of the straight lines in space, and through each point in space will pass one or more rays. Every ray will be determined by a point through which it passes which we call the initial point, and by its direction. We will take a surface cutting all the rays as the locus of the initial points, and then through each point of the surface will pass one or more rays of the system. Let $\sigma = \phi(t, u)$ be the equation of this surface, and let the direction of the ray be given by $\tau = \frac{\psi(t, u)}{T\psi(t, u)}$, where ϕ and ψ are vector functions of t and u. The locus of the extremity of τ will then be a sphere of radius unity, and when we refer to the surface τ we will mean this sphere. The equation of any ray then is of the form

$$\rho = \sigma + x\tau.$$

§1.

The Foci and Focal Planes.

Let $\rho = \sigma + x\tau$ be the equation of any ray, then any consecutive ray will be of the form

$$\rho = \sigma + d\sigma + y(\tau + d\tau).$$

In general this ray will not cut the ray τ, which we will refer to as the primitive ray. In order that it may, we have the necessary and sufficient condition,

$$d\rho = d\sigma + xd\tau + dx\tau = 0,$$

which gives us

$$S\tau d\sigma d\tau = 0, \tag{1}$$

a quadratic in dt and du; which will therefore give us two values of the ratio of dt to du. This quadratic may not always have its roots real. They will be real or imaginary according to the law which unites the straight lines into one system of rays. Therefore we have two kinds of systems of rays, one in which every ray is cut by two consecutive rays, and the other in which a ray is generally cut by no consecutive ray. As a third kind of a system, we have that in which certain rays belong to the first and others to the second kind mentioned above. The points in which consecutive rays cut the primitive ray we will call the foci of this ray, and the planes of these rays and the primitive ray we will call the focal planes. To obtain the distances of the foci from the initial point of τ we have

$$d\sigma = \partial_t\sigma\, dt + \partial_u\sigma du$$

and

$$d\tau = \partial_t\tau\, dt + \partial_u\tau du,$$

$$d\rho = d\sigma + xd\tau + dx\tau = 0$$

$$= (\partial_t\sigma + x\partial_t\tau)\, dt + (\partial_u\sigma + x\partial_u\tau)\, du + dx\tau = 0,$$

which gives us

$$S\tau\,(\partial_t\sigma + x\partial_t\tau)(\partial_u\sigma + x\partial_u\tau) = 0. \tag{2}$$

If we call the roots of (1) $\frac{dt_1}{du_1}$ and $\frac{dt_2}{du_2}$, then

$$d\tau_1 = \partial_t\tau\, dt_1 + \partial_u\tau du_1,$$
$$d\tau_2 = \partial_t\tau\, dt_2 + \partial_u\tau du_2$$

we will call the focal lines.

For the ray in question we will take

$$S\partial_u\tau\partial_t\tau = 0 \text{ and } T\partial_t\tau = T\partial_u\tau = 1.$$

Since we have two independent parameters, if $\partial_t\tau$ and $\partial_u\tau$ are not perpendicular, let us take t and u functions of two new parameters, say t' and u'.

Let

$$t = \eta\,(t', u') = a_1 + a_2t' + a_3u' + \ldots.,$$
$$u = \zeta\,(t', u') = b_1 + b_2t' + b_3u' + \ldots.$$

Then

$$dt = \eta'_{t'}dt' + \eta'_{u'}du'$$

and

$$du = \zeta'_{t'}dt' + \zeta'_{u'}du'$$

and

$$d\tau = (\eta'_{t'}\partial_t\tau + \zeta'_{t'}\partial_u\tau)\, dt' + (\eta'_{u'}\partial_t\tau + \zeta'_{u'}\partial_u\tau)\, du'.$$

We may assume any value of t' and u' for this ray, say 0, 0; for if they had the values t'_1 and u'_1 we could take $t' - t'_1$ and $u' - u'_1$ as new variables. Then for this ray we have

$$\eta'_{t'} = a_2,\ \eta'_{u'} = a_3,\ \zeta'_{t'} = b_2 \text{ and } \zeta'_{u'} = b_3,$$

since the derivatives of the terms containing t' and u' to more that the first degree would contain t' or u' and therefore vanish. We have then

$$d\tau = (a_2\partial_t\tau + b_2\partial_u\tau)\,dt' + (a_3\partial_t\tau + b_3\partial_u\tau)\,du'.$$

In any case $S\tau d\tau = 0$ since $T\tau = 1$, and therefore $\partial_t\tau$ and $\partial_u\tau$ are always perpendicular to τ. If $\partial_t\tau$ and $\partial_u\tau$ have not the same direction, since we have at our disposal four arbitrary constants, we can make

$$a_2\partial_t\tau + b_2\partial_u\tau \text{ and } a_3\partial_t\tau + b_3\partial_u\tau$$

have any directions perpendicular to τ and any lengths we please. It is evident that we can make $\partial_t\tau$ and $\partial_u\tau$ perpendicular for any ray since we have an infinity of constants at our disposal. We will so choose a_2, a_3, b_2 and b_3 that $\partial_t\tau$ and $\partial_u\tau$ will be the bisectors (of length unity) of the angles between the focal lines. We can evidently always make this choice, since if the roots of (1) are not real they are conjugate imaginaries, and hence the bisectors of the angles between the focal lines are always real. Thus we have now

$$S\partial_u\tau\partial_t\tau = 0, \quad T\partial_u\tau = T\partial_t\tau = 1$$

for the ray in question, and in every case, since the tensor of τ is constant and equal to unity, $S\tau d\tau = 0$. Let us choose also as the surface σ (the locus of initial points) the locus of the point midway between the foci on any ray. This point we will call the centre of the ray, and thus σ is the locus of centres. Let us now see what simplification these assumptions will introduce into our formulae. Since $\partial_t\tau$ and $\partial_u\tau$ are the bisectors of the angles between the focal planes, we must have the sum of the roots of (1) zero, therefore

$$S\tau\partial_t\sigma\partial_u\tau - S\tau\partial_t\tau\partial_u\sigma = 0.$$

Also, since we have taken the surface σ as the locus of centres, the sum of the roots of (2) is zero,

$$\therefore\ S\tau\partial_t\sigma\partial_u\tau + S\tau\partial_t\sigma\partial_u\sigma = 0,$$

which two equations give us

$$S\tau\partial_t\sigma\partial_u\tau = 0$$

and

$$S\tau\partial_t\tau\partial_u\sigma = 0.$$

Putting for τ, $\partial_t\tau\partial_u\tau$, we have

$$S\tau\partial_t\sigma\partial_u\tau = S\partial_t\tau\partial_u\tau\partial_t\sigma\partial_u\tau = -(\partial_u\tau)^2S\partial_t\sigma\partial_t\tau = S\partial_t\sigma\partial_t\tau \text{ since } (\partial_u\tau)^2 = -1.$$

Also

$$S\tau\partial_t\tau\partial_u\sigma = -S\partial_t\tau\tau\partial_u\sigma = -S\partial_t\tau\partial_t\tau\partial_u\tau\partial_u\sigma = -S\partial_u\sigma\partial_u\tau,$$

$$\therefore\ S\partial_t\sigma\partial_t\tau = 0, \tag{3}$$

$$S\partial_u\sigma\partial_u\tau = 0. \tag{4}$$

§2.

The Vertices of the Ray and the Principal Planes.

We will now find the distance from the initial point to the foot of the common perpendicular to τ and any consecutive ray. The direction of the common perpendicular is $\tau d\tau$. We must have

$$x\tau = d\sigma + y(\tau + d\tau) + z\tau d\tau.$$

Operating by $S(\tau + d\tau)\tau d\tau$ and neglecting terms of the third order, we have

$$x(d\tau)^2 = -Sd\sigma d\tau,$$

$$\therefore\ x = -\frac{Sd\sigma d\tau}{(d\tau)^2}$$

Since $$d\sigma = \partial_t\sigma dt + \partial_u\sigma du$$

and $$d\tau = \partial_t\tau dt + \partial_u\tau du,$$

we have $$x = \frac{dtdu\,(S\partial_u\sigma\partial_t\tau + S\partial_t\sigma\partial_u\tau)}{dt^2 + du^2} \tag{1}$$

For any value of the ratio of dt to du this expression gives the distance x of the centre of the ray from the foot of the common perpendicular to τ and the consecutive ray corresponding to this value of $\frac{dt}{du}$. By giving $\frac{dt}{du}$ all values from $-\infty$ to $+\infty$ we will get all the different values of x. For real values of dt and du the denominator of x cannot become zero, and therefore x cannot become infinite; hence it must vary between two limits, a maximum and minimum value. For a maximum or minimum value of x we must have $\frac{\delta x}{\delta \frac{dt}{du}} = 0$, and therefore we have for a maximum or minimum

$$dt^2 + du^2 - 2dt^2 = 0,$$

or $$\left(\frac{dt}{du}\right)^2 = 1,$$

$$\left(\frac{dt}{du}\right) = \pm 1. \tag{2}$$

To obtain the two values of x corresponding to these values of the ratio of $\frac{dt}{du}$, since we have $\frac{\delta x}{\delta dt} = 0$ and $\frac{\delta x}{\delta du} = 0$, we must have

$$2x dt - (S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma)\, du = 0$$

and

$$2x du - (S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma)\, dt = 0,$$

and therefore

$$\begin{vmatrix} 2x, & S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma \\ S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma, & 2x \end{vmatrix} = 0,$$

or

$$x^2 - \left(\frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2}\right)^2 = 0. \qquad (3)$$

The points so determined we will call the vertices of the ray. By taking $\frac{dt}{du} = \pm 1$ we get

$$d\tau_1 = (\partial_t\tau + \partial_u\tau)\, dt$$

and

$$d\tau_2 = (\partial_t\tau - \partial_u\tau)\, dt.$$

These lines we will call the principal lines. The planes of τ and the principal lines we will call the principal planes.

From (2) and (3) we get the following theorems:

I. The principal lines bisect the angles between $\partial_t\tau$ and $\partial_u\tau$, and therefore the principal planes are at right angles.

II. The point midway between the vertices coincides with the point midway between the foci. This point we have called the centre of the ray.

Let us call the roots of (2), §1, ρ_1 and ρ_2, and the roots of (3), §2, r_1 and r_2.

$$\rho_1 = +\sqrt{S\partial_t\tau\partial_u\sigma S\partial_u\tau\partial_t\sigma},$$

$$\rho_2 = -\sqrt{S\partial_t\tau\partial_u\sigma S\partial_u\tau\partial_t\sigma},$$

$$r_1 = \frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2},$$

$$r_2 = -\frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2},$$

we have

$$\rho_1\rho_2 - r_1 r_2 = \left(\frac{S\partial_t\sigma\partial_u\tau - S\partial_u\sigma\partial_t\tau}{2}\right)^2, \qquad (4)$$

or since

$$\rho_1 = -\rho_2 \text{ and } r_1 = -r_2,$$

$$r_1^2 - \rho_1^2 = \left(\frac{S\partial_t\sigma\partial_u\tau - S\partial_u\sigma\partial_t\tau}{2}\right)^2.$$

We have thus the theorem that the distance of the foci from the centre is never greater than the distance of the vertices from the centre. We have for r, the distance from the initial point to the foot of the common perpendicular to τ and any consecutive ray,

$$r = -\frac{Sd\sigma d\tau}{(d\tau)^2} = \frac{dtdu\,(S\partial_u\tau\partial_t\sigma + S\partial_t\tau\partial_u\sigma)}{dt^2 + du^2}.$$

Let us call ω the angle which the common perpendicular makes with the first principal line. $d\tau_1 = (\partial_t\tau + \partial_u\tau)\,dt_1$.

$$\cos^2\omega = \frac{S^2 d\tau d\tau_1}{(d\tau)^2(d\tau_1)} = -\frac{(dt+du)^2}{2\,(d\tau)^2},$$

$$\sin^2\omega = \frac{S^2 d\tau d\tau_2}{(d\tau)^2(d\tau_2)^2} = -\frac{(dt-du)^2}{2\,(d\tau)^2},$$

$$r_1\cos^2\omega = -\frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2}\,\frac{(dt+du)^2}{2\,(d\tau)^2},$$

$$r_2\sin^2\omega = \frac{S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{2}\,\frac{(dt-du)^2}{2\,(d\tau)^2},$$

$$\therefore\ r_1\cos^2\omega + r_2\sin^2\omega = -\frac{dudt\,(S\partial_t\tau\partial_u\sigma + S\partial_u\tau\partial_t\sigma}{(d\tau)^2}$$
$$= r.$$

Therefore we have the neat formula

$$r = r_1\cos^2\omega + r_2\sin^2\omega.$$

All these results have been obtained by both Hamilton and Kummer.

For the foci we have

$$d\tau_1 \parallel \partial_t\tau\sqrt{S\partial_u\sigma\partial_t\tau} + \partial_u\tau\sqrt{S\partial_u\tau\partial_t\sigma},$$
$$d\tau_2 \parallel \partial_t\tau\sqrt{S\partial_u\sigma\partial_t\tau} - \partial_u\tau\sqrt{S\partial_u\tau\partial_t\sigma};$$

for the principal lines

$$d\tau_1 = (\partial_t\tau + \partial_u\tau)\,dt_1,$$
$$d\tau_2 = (\partial_t\tau - \partial_u\tau)\,dt_2.$$

We have then for the equations of the focal planes,

$$S(\rho-\sigma)\,\tau\,(\partial_t\tau\sqrt{S\partial_u\sigma\partial_t\tau} + \partial_u\tau\sqrt{S\partial_u\tau\partial_t\sigma}) = 0$$

and

$$S(\rho-\sigma)\,\tau\,(\partial_t\tau\sqrt{S\partial_u\sigma\partial_t\tau} - \partial_u\tau\sqrt{S\partial_u\tau\partial_t\sigma}) = 0.$$

Putting $\tau = \partial_u\tau\partial_t\tau$, we have for these equations

$$S(\rho-\sigma)\{\partial_t\tau\sqrt{S\partial_u\tau\partial_t\sigma} - \partial_u\tau\sqrt{S\partial_u\sigma\partial_t\tau}\} = 0, \qquad (6)$$

and

$$S(\rho-\sigma)\{\partial_t\tau\sqrt{S\partial_u\tau\partial_t\sigma} + \partial\ \tau\sqrt{S\partial_u\sigma\partial_t\tau}\} = 0. \qquad (7)$$

For the principal planes we have

$$S(\rho-\sigma)(\partial_t\tau-\partial_u\tau)=0, \tag{8}$$

$$S(\rho-\sigma)(\partial_t\tau+\partial_u\tau)=0. \tag{9}$$

These equations have not been given by Kummer.

§3.

The Surfaces connected with the System of Rays.

The foot-points on every ray of the system, namely, the two foci, the vertices and the common middle point, determine five surfaces as their loci. The two surfaces on which the vertices lie present themselves analytically as but one surface, in so far as they are both represented by one equation, but they may be looked upon as two surfaces or the two parts of one surface. Let us call the two surfaces F_1 and F_2. These two surfaces so divide space that between them lie the feet of the common perpendiculars to any ray and all its consecutive rays, but outside of them none.

When we have taken the surface σ as the locus of centres and not making the assumption $T\partial_t\tau=T\partial\,\tau=1$, we have the equations of the principal surfaces,

$$\rho=\sigma+\tau\sqrt{\frac{(S\partial_t\tau\partial_u\sigma-S\partial_u\tau\partial_t\sigma)^2-4S(V\partial_u\tau\partial_t\tau)\,\partial_t\sigma\partial_u\sigma}{4S(V\partial_u\tau\partial_t\tau)\,\partial_t\tau\partial_u\tau}}, \tag{1}$$

$$\rho=\sigma-\tau\sqrt{\frac{(S\partial_t\tau\partial_u\sigma-S\partial_u\tau\partial_t\sigma)^2-4S(V\partial_u\tau\partial_t\tau)\,\partial_t\sigma\partial_u\sigma}{4S(V\partial_u\tau\partial_t\tau)\,\partial_t\tau\partial_u\tau}}. \tag{2}$$

The two surfaces on which the foci of every ray lie, which we call the focal surfaces and which we will designate by ϕ_1 and ϕ_2, exist as real surfaces only when the roots of (1), §1 are real. Their equations are

$$\rho=\sigma+\tau\sqrt{\frac{S\tau\partial_u\sigma\partial_t\sigma}{S\tau\partial_t\tau\partial_u\tau}} \tag{3}$$

$$\rho=\sigma-\tau\sqrt{\frac{S\tau\partial_u\sigma\partial_t\sigma}{S\tau\partial_t\tau\partial_u\tau}}.$$

We may now find what relation the rays have to this surface. For the tangent plane to ϕ_1 at the first focus whose distance from the centre is ρ_1,

$$d\rho=\left(\partial_t\sigma+\rho_1\partial_t\tau+\tau\frac{\partial\rho_1}{\partial t}\right)dt+\left(\partial_u\sigma+\rho_1\partial_u\tau+\tau\frac{\partial\rho_1}{\partial u}\right)du,$$

$$\begin{aligned}S\tau\partial_t\rho\partial_u\rho&=S\tau\left(\partial_t\sigma+\rho_1\partial_t\tau+\tau\frac{\partial\rho_1}{\partial t}\right)\left(\partial_u\sigma+\rho_1\partial_u\tau+\tau\frac{\partial\rho_1}{\partial u}\right)\\&=S\tau\,(\partial_t\sigma+\rho_1\partial_t\tau)(\partial_u\sigma+\rho_1\partial_u\tau);\end{aligned}$$

but since ρ_1 is a root of

$$S\tau\,(\partial_t\sigma + x\partial_t\tau)(\partial_u\sigma + x\partial_u\tau) = 0$$

we have

$$S\tau\partial_t\rho\partial_u\rho = 0; \tag{5}$$

∴ every ray touches each focal surface, and the system of rays may be therefore looked upon as a system of common tangents to two surfaces, or a system of double tangents to a single surface. If the two surfaces ϕ_1 and ϕ_2 intersect, then every straight line tangent to the curve of intersection will be a ray of the system since it touches each surface once, namely, in the point at which it is tangent to the curve. The middle points lie on a surface whose equation is

$$\rho = \sigma + \frac{\rho_1 + \rho_2}{2}\tau,$$

or

$$\rho = \sigma + \frac{S\tau\,(\partial_t\tau\partial_u\sigma + \partial_t\sigma\partial_u\tau)}{2S\tau\partial_u\tau\partial_t\tau}\tau. \tag{6}$$

This surface is always real.

All these surfaces may in special cases degenerate into lines or even into points. As a special case of a system of rays we may have all the straight lines through a point, in which case the focal surface is nothing but a point. Also the rays may be normals to a surface, in which case, as we shall see afterwards, the focal and principal surfaces coincide.

§4.

The Measure of Density.

Since $T\tau = 1$, the locus of the extremity of τ is a sphere of radius unity. Then for each ray of the system we will have a corresponding point on the sphere, and for each continuous succession of rays a continuous curve on the sphere. If at any point of a ray we pass a plane perpendicular to the ray and take a closed curve in this plane, then to the group of rays which pass through this curve we will have a corresponding closed curve on the sphere. If we take the closed curve so that its points are at an infinitesimal distance from the ray, then we will have a corresponding infinitesimal curve on the sphere. The ratio of the area of the curve on the sphere to that of the curve in the plane (which in the case when the system of rays is normal to a surface, and the perpendicular plane a tangent plane, has been called by Gauss the measure of absolute curvature) we will call the measure of density. If at any point of a ray we pass a plane through it perpendicular to the ray, which will cut the bundle of consecutive rays in a curve whose area we will call f, and the area of the corresponding curve on the sphere we call ϕ, then the ratio $\frac{\phi}{f}$ is the measure of density of the system of rays at this point. We have

$$\phi = \frac{1}{2}\int TV.d\tau\,(d\tau + \delta d\tau) = \frac{1}{2}\int TV.d\tau\delta d\tau,$$

where $d\tau = \partial_t\tau dt + \partial_u\tau du$

and $\delta d\tau = \partial_t\tau\delta dt + \partial_u\tau\delta du$,

$$\therefore\ \phi = \frac{1}{2}\int (dt\delta du - du\delta dt). \tag{1}$$

For the perpendicular $d\sigma + x\,(\tau + d\tau)$ on τ we have

$$= \tau V\tau\{d\sigma + x\tau + xd\tau\}$$
$$= -xd\tau + \tau V\tau d\sigma.$$

Then for the equation of the section of the bundle of rays by a plane perpendicular to τ and distant x from the centre we have

$$\rho = xd\tau + \tau Vd\sigma\tau, \tag{2}$$

$$\therefore\ f = \frac{1}{2}\int TV.(xd\tau + \tau Vd\sigma\tau)(x\delta d\tau + \tau V\delta d\sigma\tau)\,xd\tau + \tau Vd\sigma\tau$$
$$= \partial_t\tau\,(xdt - duS\partial_t\tau\partial_u\sigma) + \partial_u\tau\,(xdu - dtS\partial_u\tau\partial_t\sigma).$$

Let us write

$$S\partial_t\tau\partial_u\sigma \equiv B \text{ and } S\partial_u\tau\partial_t\sigma \equiv C.$$

Then

$$f = \frac{1}{2}\int (xdt - Bdu)(x\delta du - C\delta dt) - (xdu - Cdt)(x\delta dt - B\delta du)$$
$$= \frac{1}{2}\int (x^2 - BC)(dt\delta du - du\delta dt), \tag{3}$$

$$\therefore\ f = (x^2 - BC)\,\phi,$$

or $\frac{\phi}{f} = \frac{1}{x^2 - BC}$,

but $\rho_1\rho_2 = -BC$,

we have $$\frac{\phi}{f} = \frac{1}{(\rho_1 - x)(\rho_2 - x)} \text{ since } \rho_1 + \rho_2 = 0. \tag{4}$$

Therefore we get the following theorem: The measure of density at any point of a ray is equal to the reciprocal of the product of the distances of this point from the foci of the ray. The measure of density of a ray at any point is always real even if the foci are imaginary. For points lying outside of the focal surfaces the measure of density is always positive, while for points inside it is always negative. We can easily see from the expression that it has its least negative value for $x = 0$ or at the middle point, since $\rho_1\rho_2$ is negative, and at the foci it is infinitely great. In the system of rays with imaginary foci the measure of density is always positive, since $\rho_1\rho_2$ is positive, and is a maximum for $x = 0$, *i. e.* in the middle point. If we now take a section of the bundle of rays at another point x', we have for the measure of density at this point

$$\vartheta' = \frac{\Phi}{f'},$$

$$\therefore \frac{f}{f'} = \frac{\vartheta'}{\vartheta}.$$

We thus have the theorem that the area of a section of a bundle of rays at any point (the plane of the section being perpendicular to τ) varies inversely as the measure of density. All the points on the different rays of the system which have the same value of the measure of density, lie on a surface which we will call a surface of equal measure of density. Since the measure of density has all possible values, we will have an infinite number of such surfaces. For the equation of a surface with the constant measure of density ϑ, we have

$$x^2 + \rho_1\rho_2 = \frac{1}{\vartheta},$$

$$x = \pm\sqrt{\frac{1}{\vartheta} - \rho_1\rho_2},$$

therefore the equation is

$$\rho = \sigma \pm \tau\sqrt{\frac{1}{\vartheta} - \rho_1\rho_2}.$$

In order that these surfaces be real it is necessary and sufficient that $\frac{1}{\vartheta}$ should lie between $\rho_1\rho_2$ and ∞. For $\vartheta = \infty$ the surface will be real if the foci are real. We see then that the focal surfaces are surfaces of the constant measure of density ∞. The construction of these surfaces is very easy when the focal surfaces are given.

§5.

The Twist of a Ray.

If we are given two lines in space, and as we pass along the second drop perpendiculars from each point to the first, then we will call the angle passed through by the perpendicular from a point a to a point b in the first ray the twist of the second line about the first from a to b.

For consecutive rays the twist for the length of the whole line is two right angles. If the two lines lie in a plane, the twist of the second about the first is zero or two right angles according as the segment of the first does not or does contain the point of intersection.

We have for the perpendicular from $d\sigma + x(\tau + d\tau)$ to τ by §4, eq. (2),

$$d\rho = xd\tau + \tau V d\sigma\tau;$$

at the initial point $x = 0$,

$$d\rho_0 = \tau V d\sigma\tau.$$

For brevity write $B \equiv S\partial_t\tau\partial_u\sigma$, $C \equiv S\partial_u\tau\partial_t\sigma$. Call α_0 the angle made by $d\rho_0$ with the principal line $d\tau_1 = (\partial_t\tau + \partial_u\tau)\,dt$,

$$\cos\,2\alpha_0 = \cos^2\alpha_0 - \sin^2\alpha_0 = \frac{S^2 d\tau_1\tau\,Vd\sigma\tau}{(d\tau_1)^2(\tau\,V\tau d\sigma)^2} - \frac{S^2 d\tau_2\tau\,Vd\sigma\tau}{(d\tau_2)^2(\tau\,Vd\sigma\tau)^2},$$

where $d\tau_2$ is the other principal line $(\partial_t\tau - \partial_u\tau)\,dt_2$,

$$d\rho = \partial_t\tau\,(xdt - Bdu) + \partial_u\tau\,(xdu - Cdt),$$

$$d\rho_0 = -Bdu\partial_t\tau - Cdt\partial_u\tau,$$

$$\frac{S^2 d\tau_1\tau\,Vd\sigma\tau}{(d\tau_1)^2(\tau\,Vd\sigma\tau)^2} = \frac{(Cdt + Bdu)^2}{2\,(B^2du^2 + C^2dt^2)},$$

$$\frac{S^2 d\tau_2\tau\,Vd\sigma\tau}{(d\tau_2)^2(\tau\,Vd\sigma\tau)^2} = \frac{(Cdt - Bdu)^2}{2\,(B^2du^2 + C^2dt^2)},$$

$$\cos\,2\alpha_0 = \frac{2BCdudt}{B^2du^2 + C^2dt^2}, \tag{1}$$

$$\sin\,2\alpha_0 = \frac{C^2dt^2 - B^2du^2}{B^2du^2 + C^2dt^2}. \tag{2}$$

Let us call β the twist of the second ray about the first when we advance from the initial point a distance x along the first ray, then

$$\tan\,\beta = -\frac{TVd\rho d\rho_0}{Sd\rho d\rho_0} = \frac{x\,(Cdt^2 - Bdu^2)}{B^2du^2 + C^2dt^2 - x\,(B + C)\,dudt},$$

$$x = \frac{(B^2du^2 + C^2dt^2)\,\sin\beta}{(Cdt^2 - Bdu^2)\,\cos\beta + (B + C)\,dudt\,\sin\beta}. \tag{3}$$

Multiply both numerator and denominator by $\dfrac{2BC}{C^2dt^2 + B^2du^2}$ and add to and subtract from the denominator $\frac{C^3dt^2 - B^3du^2}{B^2du^2 + C^2dt^2}\cos\beta$. We get

$$x = \frac{2BC\sin\beta}{\frac{(B-C)(B^2du^2+C^2dt^2)\cos\beta + (B+C)\}(C^2dt^2 - B^2du^2)\cos\beta + 2BCdudt\sin\beta\{}{B^2du^2+C^2dt^2}}$$

$$\therefore x = \frac{2BC\sin\beta}{(B-C)\cos\beta + (B+C)\sin(2\alpha_0+\beta)}. \tag{4}$$

Let us regard β as a constant angle and inquire along what rays we get a maximum or a minimum value of x. The maximum and minimum values are evidently given by

$$\sin(2\alpha_0+\beta) = 1 \text{ and } \sin(2\alpha_0+\beta) = -1.$$

Let us put

$$x_1 = \frac{2BC\sin\beta}{(B-C)\cos\beta - (B+C)}, \quad x_2 = \frac{2\beta C\sin\beta}{(B-C)\cos\beta + (B+C)}.$$

Calling α the angle made by $d\rho$ with $\partial_t\tau + \partial_u\tau$, we have

$$\beta = \alpha - \alpha_0,$$

$$\frac{1}{x} - \frac{1}{x_1} = \frac{2\cos^2\left(\alpha_0 + \frac{1}{2}\beta - \frac{\pi}{4}\right)(B+C),}{2BC\sin\beta} \tag{5}$$

$$\frac{1}{x_2} - \frac{1}{x} = \frac{2\sin^2\left(\alpha_0 + \frac{\beta}{2} - \frac{\pi}{4}\right)(B+C),}{2BC\sin\beta}, \tag{6}$$

$$\frac{1}{x} = \frac{\cos^2\left(\alpha_0 + \frac{\beta}{2} - \frac{\pi}{4}\right)}{x_2} + \frac{\sin^2\left(\alpha_0 + \frac{\beta}{2} - \frac{\pi}{4}\right)}{x_1}, \tag{7}$$

Thus if we proceed along any ray until its twist about the primitive ray is equal to a constant angle β, this equation gives us the relation between the distance along the primitive ray for this ray and the angle made by this ray with the principal line $d\tau$ and the maximum and minimum distances. This relation becomes, when the rays are normal to a surface, the Eulerian theorem,

$$\frac{1}{R} = \frac{\cos^2\alpha_0}{R_2} + \frac{\sin^2\alpha_0}{R_1},$$

where R_1 and R_2 are the radii of curvature in the principal section, and R the radius of curvature in any normal plane, and α_0 the angle made by this plane with one of the principal planes. In the case of the two focal rays the twist is either zero or two right angles. In the case where the foci are imaginary, the twist cannot be zero for a finite segment of the line, and therefore the twist of the ray cannot change its sense. Thus we have two kinds of systems of rays with imaginary foci, those in which the twist is to the right and those in which it is to the left. When the foci are real, the twist of any ray from one focus to the other is found as follows:

$$d\rho = \partial_t\tau\,(xdt - Bdu) + \partial_u\tau\,(xdu - Cdt)$$

for the foci
$$x = \pm\sqrt{BC} \equiv \pm A,$$

then
$$d\rho_1 = \partial_t\tau\,(Adt - Bdu) + \partial_u\tau\,(Adu - Cdt), \tag{8}$$
$$-d\rho_2 = \partial_t\tau\,(Adt + Bdu) + \partial_u\tau\,(Adu + Cdt). \tag{9}$$

Calling β the angle between $d\rho_1$ and $d\rho_2$, we have

$$\tan^2\beta = -\frac{V^2 d\rho_1 d\rho_2}{S^2 d\rho_1 d\rho_2},$$
$$V^2 d\rho_1 d\rho_2 = 4A^2(Bdu^2 - Cdt^2)^2,$$
$$S^2 d\rho_1 d\rho_2 = (B - C)^2(Bdu^2 - Cdt^2)^2,$$
$$\therefore\ \tan^2\beta = \frac{4A^2}{(B - C)^2},$$

and is therefore independent of $\dfrac{dt}{du}$.

We have the focal lines

$$d\tau_1 \,\|\, \sqrt{B}\,\partial_t\tau - \sqrt{C}\,\partial_u\tau,$$
$$d\tau_2 \,\|\, \sqrt{B}\,\partial_t\tau + \sqrt{C}\,\partial_u\tau.$$

Calling β_0 the angle between them, we have

$$\tan^2\beta_0 = -\frac{V^2 d\tau_1 d\tau_2}{S^2 d\tau_1 d\tau_2} = \frac{4BC}{(B - C)^2}.$$

Since $A^2 = BC$, we have

$$\beta = \beta_0. \tag{10}$$

Therefore we have the following theorem: The twist of any consecutive ray from focus to focus of the primitive ray is constant and equal to the angle between the focal lines.

§6.

The Sections of the Bundle of Rays.

If we construct at any point of a ray a plane perpendicular to it, we get an infinitesimal curve lying in this plane which is the section of the bundle of consecutive rays. The vector equation of this curve is

$$\rho = xd\tau + \tau V d\sigma\tau,$$

or
$$\rho = \partial_t\tau\,(xdt - Bdu) + \partial_u\tau\,(xdu - Cdt),$$

the origin being the point at which the perpendicular plane is constructed. To see the character of the sections at the foci, we put $x = \pm A$. Writing $x = A$ we have

$$\rho = \partial_t \tau (A dt - B du) + \partial_u \tau (A du - C dt);$$

but $\frac{A}{B} = \frac{C}{A}$ since $A^2 = BC$,

$$\therefore \rho = \left(\frac{A}{B} dt - du\right)(B \partial_t \tau - A \partial_u \tau)$$

$$= \sqrt{B}\left(\frac{A}{B} dt - du\right)(\sqrt{B} \partial_t \tau - \sqrt{C} \partial_u \tau).$$

But the focal line $d\tau_1 \| \sqrt{B} \partial_t \tau - \sqrt{C} \partial_u \tau$. Therefore we have the following theorem: The sections at the foci are lines coinciding in direction with the focal lines. The lengths of these lines are zero when

$$\sqrt{B} \partial_t \tau - \sqrt{C} \partial_u \tau = 0,$$

and

$$\sqrt{B} \partial_t \tau + \sqrt{C} \partial_u \tau = 0,$$

$$\therefore B = 0, \quad C = 0.$$

But since

$$r_1 r_2 = -\left(\frac{S \partial_t \tau \partial_u \sigma + S \partial_u \tau \partial_t \sigma}{2}\right)^2$$

and

$$\rho_1 \rho_2 = -BC,$$

we have

$$r_1 = r_2 = \rho_1 = \rho_2 = 0,$$

and therefore in this case the two foci and the two vertices coincide with the centre. Thus we have every consecutive ray going through the centre of the primitive ray. These rays we will call principal rays. Generally there are no principal rays since we have three equations to be satisfied, namely, all the coefficients of §1, eq. (1) must be zero. These conditions are in general only equivalent to two; but there is also another condition, namely, that the foci must be real or

$$S \partial_t \tau \partial_u \sigma = S \partial_u \tau \partial_t \sigma.$$

In case only the two foci coincide with the centre, the section of the bundle of rays is only a straight line at the centre. The section in this case would lie in the plane in which the two focal planes have united. Because the condition that the two foci coincide with the centre only involves one equation between the parameters, there is generally a continuous succession of such lines which form a ruled surface. All rays which are tangent to the curve of intersection of the two focal surfaces have their foci coinciding.

Let us now investigate the character of these sections. Suppose dt and du are connected by the equation

$$B^2du^2 + C^2dt^2 = \varkappa^2,$$

where $\varkappa$ is an infinitesimal constant. We have

$$d\rho = (xdt - Bdu)\,\partial_t\tau + (xdu - Cdt)\,\partial_u\tau .$$

Since $\partial_t\tau$ and $\partial_u\tau$ are unit vectors perpendicular to each other, we may write

$$xdt - Bdu \equiv X,$$

and

$$xdu - Cdt \equiv Y,$$

then

$$dt = \frac{xX + BY}{x^2 - BC} \text{ and } du = \frac{xY + CX}{x^2 - BC} .$$

Therefore the Cartesian equation of any section is

$$C^2\left(\frac{xX + BY}{x^2 - BC}\right)^2 + B^2\left(\frac{xY + CX}{x^2 - BC}\right)^2 = \varkappa^2.$$

Thus we see that under these conditions the section of the bundle of rays is always an ellipse with its centre in τ.

For $x = 0$ this is a circle. Therefore the perpendicular section at the centre of the ray is a circle of radius $\varkappa$.

If we call ϑ the angle made by the major axis on any section with X, we have

$$\tan 2\vartheta = \frac{2BC}{x(C - B)},$$

or, since $\rho_1\rho_2 = -BC$,

$$\tan 2\vartheta = \frac{2\rho_1\rho_2}{x(B - C)};$$

for $x = \rho_1$,

$$\tan 2\vartheta = \frac{2\rho_2}{(B - C)},$$

for $x = \rho_2$,

$$\tan 2\vartheta = \frac{2\rho_1}{(B - C)}.$$

These give us the tangent of double the angle made by the focal lines with $\partial_t\tau$. As we shall see further on, in the case when the rays are normal to a surface, $B = C$,

$$\therefore \tan 2\vartheta = \infty,$$

$$\vartheta = 45^\circ,$$

$\therefore$ the principal lines are the principal axes of all sections in this case. Let us now obtain the expressions for the lengths of the axes in any case. When the equation of any ellipse is in the form

$$a_{11}x^2 + a_{22}y^2 + a_{33} + 2a_{12}xy + 2a_{13}x + 2a_{23}y = 0,$$

the squares of the semi-axes are given by $-\frac{A}{\lambda_1 A_{33}}$ and $\frac{A}{-\lambda_2 A_{33}}$, where

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}$$

$a_{13} = a_{31}$, etc., and λ_1 and λ_2 are the roots of

$$\lambda^2 - (a_{11} + a_{12})\,\lambda + a_{11}a_{22} - a_{12}^2 = 0.$$

In our case we get

$$\lambda^2 - \{(B^2 + C^2)x^2 + 2B^2C^2\}\,\lambda + x^2(x^2 - BC)^2 = 0,$$

$$\frac{A}{A_{33}} = -x^2(x^2 - BC)^2,$$

we have $$\lambda = \frac{(B^2 + C^2)x^2 + 2B^2C^2 \pm x(B + C)\sqrt{x^2(B - C)^2 + 4B^2C^2}}{2}.$$

If we write

$$2\lambda_1 \equiv (B^2 + C^2)x^2 + 2B^2C^2 - x(B + C)\sqrt{x^2(B - C)^2 + 4B^2C^2},$$

and $$2\lambda_2 \equiv (B^2 + C^2)x^2 + 2B^2C^2 + x(B + C)\sqrt{x^2(B - C)^2 + 4B^2C^2},$$

for the first focus $x = \sqrt{BC}$.

Substituting this value of x in the expressions for the two axes, we have

$$-\frac{A}{\lambda_2 A_{33}} = 0.$$

We have

$$-\frac{A}{\lambda_1 A_{33}} = \frac{2x^2(x^2 - BC)^2}{(B^2 + C^2)x^2 + 2B^2C^2 - x(B + C)\sqrt{x^2(B - C)^2 + 4B^2C^2}},$$

which is indeterminate, being of form $\frac{0}{0}$.

Differentiating numerator and denominator twice, and substituting $x = \sqrt{BC}$, we obtain $$-\frac{A}{\lambda_1 A_{33}} = \frac{x^2(B + C)^2}{BC},$$

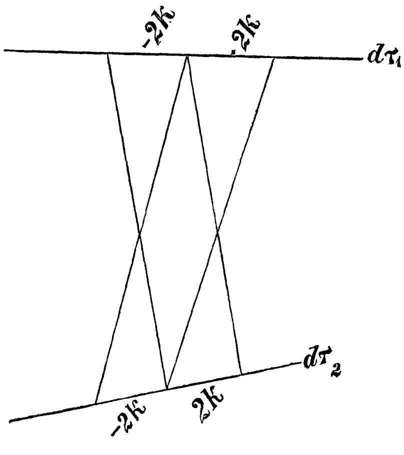
or the length of the major axis is $\frac{2x(B + C)}{\sqrt{BC}}$, the length of the minor axis being zero. The lengths of the sections at the foci are therefore in this case, viz. when the section at the centre is a circle of radius $\varkappa$,

$$\frac{2\varkappa(B+C)}{\rho_1} \text{ and } \frac{2\varkappa(B+C)}{\rho_2},$$

we have
$$d\rho = \partial_t\tau\,(xdt - Bdu) + \partial_u\tau\,(xdu - Cdt)\,;$$
at the foci $x = \pm\sqrt{BC}$,
$$\therefore\ d\tau_1 = (\sqrt{C}dt - \sqrt{B}du)\{\sqrt{B}\partial_t\tau - \sqrt{C}\partial_u\tau\},$$
$$d\tau_2 = (\sqrt{C}dt + \sqrt{B}du)\{\sqrt{B}\partial_t\tau + \sqrt{C}\partial_u\tau\}.$$
These are the focal lines, and it is interesting to observe that through each point on the focal line two rays pass, namely, the two rays corresponding to the values of dt and du which we get from the two equations

$$\sqrt{C}dt - \sqrt{B}du = \text{constant},$$
or
$$\sqrt{C}dt + \sqrt{B}du = \text{ ``}$$
and
$$B^2du^2 + C^2dt^2 = \varkappa^2\,;$$
for a ray passing through the first focus
$$\sqrt{C}dt - \sqrt{B}du = 0,$$
or
$$\frac{dt}{du} = \frac{\sqrt{B}}{\sqrt{C}}.$$
Substituting in $B^2du^2 + C^2dt^2 = \varkappa^2$, we get the two pairs of values

$$dt = \frac{\varkappa}{\sqrt{C(B+C)}},\quad du = \frac{\varkappa}{\sqrt{B(B+C)}}$$
and
$$dt = -\frac{\varkappa}{\sqrt{C(B+C)}},\quad du = -\frac{\varkappa}{\sqrt{B(B+C)}}.$$
We may inquire where these rays cut the second focal line $d\tau_2$. Substituting in $d\tau_2$ we get the lengths to lie $2\varkappa$ and $-2\varkappa$. The case is then thus:



To obtain the values of dt and du which give us the rays passing through the extremities of the focal lines, we have for $d\tau_1$

$$\sqrt{C}\delta dt - \sqrt{B}\delta du = 0$$

and $B^2 du\delta du + C^2 dt\delta dt = 0$,

$$\therefore \frac{dt}{du} = -\frac{B\sqrt{B}}{C\sqrt{C}}.$$

Substituting in $B^2du^2 + C^2dt^2 = \kappa^2$, we get

$$dt = -\frac{\sqrt{B}\kappa}{C\sqrt{B+C}},\ du = \frac{\kappa\sqrt{C}}{B\sqrt{B+C}}$$

and

$$dt = +\frac{\sqrt{B}\kappa}{C\sqrt{B+C}},\ du = -\frac{\kappa\sqrt{C}}{B\sqrt{B+C}}.$$

For the other focal line

$$\sqrt{C}\delta dt + \sqrt{B}\delta du = 0$$

and

$$B^2 du\delta du + C^2 dt\delta dt = 0,$$

$$\therefore \frac{dt}{du} = \frac{B\sqrt{B}}{C\sqrt{C}},$$

and substituting in $B^2du^2 + C^2dt^2 = \kappa^2$, we get

$$dt = \frac{\kappa\sqrt{B}}{C\sqrt{B+C}},\ du = \frac{\kappa\sqrt{C}}{B\sqrt{B+C}}$$

and

$$dt = -\frac{\kappa\sqrt{B}}{C\sqrt{B+C}},\ du = -\frac{\kappa\sqrt{C}}{B\sqrt{B+C}}.$$

We thus see that the rays passing through the extremities of the focal lines do not coincide. Let us now find where the ray

$$dt = \frac{\kappa\sqrt{B}}{C\sqrt{B+C}},\ du = -\frac{\kappa\sqrt{C}}{B\sqrt{B+C}},$$

passing through the extremity of $d\tau_1$, cuts $d\tau_2$. We have for the distance of this point from the second focus

$$\sqrt{B+C}\,(\sqrt{C}dt + \sqrt{B}du).$$

In this case we have this distance

$$\sqrt{B+C}\left\{\frac{\kappa\sqrt{B}}{\sqrt{C(B+C)}} - \frac{\sqrt{C}\kappa}{\sqrt{B(B+C)}}\right\} = \kappa\frac{B-C}{\sqrt{BC}}.$$

For the principal rays $$\frac{dt}{du} = \pm 1.$$

Substituting in $B^2du^2 + C^2dt^2 = \varkappa^2$, we get for the four rays

$$dt = \frac{\varkappa}{\sqrt{B^2+C^2}}, \quad du = \frac{\varkappa}{\sqrt{B^2+C^2}}, \qquad \text{I.}$$

$$dt = -\frac{\varkappa}{\sqrt{B^2+C^2}}, \quad du = -\frac{\varkappa}{\sqrt{B^2+C^2}}, \qquad \text{II.}$$

$$dt = \frac{\varkappa}{\sqrt{B^2+C^2}}, \quad du = -\frac{\varkappa}{\sqrt{B^2+C^2}}, \qquad \text{III.}$$

$$dt = -\frac{\varkappa}{\sqrt{B^2+C^2}}, \quad du = \frac{\varkappa}{\sqrt{B^2+C^2}}. \qquad \text{IV.}$$

For the points where I cuts the focal lines we have

$$\sqrt{B+C}\{\sqrt{C}dt - \sqrt{B}du\}$$

and $$\sqrt{B+C}\{\sqrt{C}dt + \sqrt{B}du\},$$

and substituting, we have

$$\frac{\varkappa\sqrt{B+C}}{\sqrt{B^2+C^2}}(\sqrt{C} - \sqrt{B}) \text{ and } \frac{\varkappa\sqrt{B+C}}{\sqrt{B^2+C^2}}\{\sqrt{C} + \sqrt{B}\}.$$

For the ratio of the lengths cut off on the focal lines by any ray we have, calling δ that on one and ε that on other,

$$\frac{\delta}{\varepsilon} = \frac{\sqrt{C}dt - \sqrt{B}du}{\sqrt{C}dt + \sqrt{B}du}.$$

§7.

Systems of Rays Normal to a Surface.

Let $\rho = \sigma + x\tau$, where x is a function of t and u, be the equation of any surface to which the rays are normal. Then we must have

$$S\tau\partial_t\rho = 0 \text{ and } S\tau\partial_u\rho = 0,$$

$$\partial_t\rho = \partial_t\sigma + x\partial_t\tau + \frac{\partial x}{\partial t}\tau,$$

$$\partial_u\rho = \partial_u\sigma + x\partial_u\tau + \frac{\partial x}{\partial u}\tau,$$

$$S\tau\partial_t\rho = S\tau\partial_t\sigma - \frac{\partial x}{\partial t} = 0, \text{ since } \tau^2 = -1 \text{ and } S\tau\partial_t\tau = 0,$$

$$S\tau\partial_u\rho = S\tau\partial_u\sigma - \frac{\partial x}{\partial u} = 0,$$

∴ we must have

$$\frac{\partial}{\partial t} S\tau\partial_u\sigma = \frac{\partial}{\partial u} S\tau\partial_t\sigma, \text{ since } \frac{\partial^2 x}{\partial t\partial u} = \frac{\partial^2 x}{\partial u\partial t},$$

∴ we have as a necessary and sufficient condition

$$S\partial_t\tau\partial_u\sigma = S\partial_u\tau\partial_t\sigma. \tag{1}$$

When this condition is satisfied, we have of course a whole family of surfaces normal to the rays, since this is only one equation of condition between t and u. When $S\partial_t\tau\partial_u\sigma = S\partial_u\tau\partial_t\sigma$, the roots of equation (1), §1 are the same as those of (2), §2, and also those of (2), §1 the same as those of (3), §2. Therefore, in those systems whose rays are normals to a surface, the focal and principal planes coincide, as do also the foci and vertices. If in this case we choose any surface which cuts the rays normally, we have the distances of the foci from the initial point equal to the two principal radii of curvature, and also have the theorem

$$r = r_1 \cos^2\omega + r_2 \sin^2\omega.$$

If we take the bundle of normals consecutive to the given normal, then we have seen that the perpendicular sections at the foci are in the direction of the focal lines. Therefore all the normals consecutive to a given normal pass through two lines, one perpendicular to the normal at one principal centre of curvature and the other at the other, and also since in this case the focal lines coincide with the principal lines, these two lines lie in the planes of principal section. In the case we are considering the principal and focal surfaces coincide and become the surfaces treated by Monge, namely, the loci of the principal centres of curvature. For the lines of curvature of the normal surface we have

$$S\tau d\tau d\rho = 0,$$

since τ is normal to the surface and this is the condition that a consecutive normal cut the primitive one, but

$$d\rho = d\sigma + x_1 d\tau + dx_1\tau,$$

∴ condition is $S\tau d\sigma d\tau = 0$, and ∴ we have the theorem that the focal lines give the directions of the lines of curvature on the surface. The umbilics of a surface, or the points at which the normal is cut by all consecutive normal and consequently the two principal centres of curvature coincide, correspond to the initial point on a principal ray in the general theory. Also in the case considered the measure of density coincides exactly with Gauss' measure of absolute curvature. Our formula for the measure of density becomes the reciprocal of the product of the distance of the point from the foci; in the case of normals, the measure of density at the initial point is the reciprocal of the product of the principal radii of curvature.

In conclusion I would state that the arrangement of this paper is like that of Kummer, and also that I am indebted to Dr. Story of the Johns Hopkins University for many valuable suggestions.